Immersed Boundary Method implementation of the turbulent boundary layer at the ice shelf/ocean interface

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Abstract

This document presents an algorithm for computing the fluid forcing required to impose the boundary conditions associate with the turbulent planetary boundary layer near the ice shelf/ocean interface. The implementation is indended for use in an Immersed Boundary Method (IBM) implemented in the Parallel Ocean Program (POP), and is intended for use in the near future in coupling the physics between POP and Glimmer: The Community Ice Sheet Model (Glimmer-CISM). The analytic solutions for the variation of the velocity and the active tracer quantities (temperature and salinity) in the boundary layer.

1 Nomenclature

Symbol	**Description**
$a =$	$(1/R_c + 1/\mu_* \xi_N)(1-\eta*)\,\mu_* \eta_*$
a_T	proportionality between salinity and temperature in linear freezing point relaiton
$b(x,y)$	the (positive) height field representing the ice/ocean interface
b_T	temperature offset in linear freezing point relaiton
C_D	non-dimensional drag coefficient
$c_{p,i}$	specific heat capacity of ice
$c_{p,o}$	specific heat capacity of ocean
c_T	proportionality between pressure and temperature in linear freezing point relaiton
f	magnitude of Corilois parameter
g	acceleration of gravity
$H(x,y)$	the (positive) depth of the bathymetry below sea level
$i =$	$\sqrt{-1}$
K	eddy viscosity

Symbol		**Description**
K_h		turbulent diffusivity of heat
K_S		turbulent diffusivity of salt
k	$=$	$0.4,$ von Karman's constant
L	$=$	$\rho_0 u_*^3 / \left[gk\left(\gamma_S \left\langle u_z' S' \right\rangle_0 - \gamma_T \left\langle u_z' T' \right\rangle_0\right)\right]$, Obukhov length
L_f		latent heat of fusion
n		distance from interface (positive into ice)
n_0		surface roughness
n_{bl}	$=$	$-ku_*\eta_*/f,$ edge of the boundary layer
n_{ref}		reference distance from interface
n_{sl}		$-u_*\eta_*^2\xi_N/f,$ edge of the surface layer
p_0		pressure at interface (depth dependent)
R_c	$=$	$0.2,$ critical Richardson number
S		salinity
S_0		salinity at interface
T		temperature
T_0		temperature at interface
$\hat{\boldsymbol{u}}$	$=$	$(\boldsymbol{u}_t - \boldsymbol{u}_{t,\infty})/\boldsymbol{u}_*,$ complex non-dimensional tangential velocity deviation
$\hat{\boldsymbol{u}}_0$		complex non-dimensional tangential surface velocity deviation
u_n		normal velocity
$u_{n,0}$		normal velocity at interface, melt rate in ocean
$u_{n,0,i}$		melt rate of ice
$\boldsymbol{u}_t$		complex tangential velocity
$\boldsymbol{u}_{t,\infty}$		complex tangential velocity outside the boundary layer
$\boldsymbol{u}_*$		complex surface friction velocity
δ	$=$	$(\pm i/k\xi_N)^{1/2},$ complex attenuation coefficient
η_*	$=$	$(1+\xi_N\mu_*/(R_c fL)^{1/2},$ stability parameter
κ_i^T		molecular diffusivity heat in ice
κ_o^S		molecular diffusivity salt in ocean water
κ_o^T		molecular diffusivity heat in ocean water
λ		one of $\{T,S\}$
μ_*	$=$	$u_*/(fL)$
ν		molecular viscosity of ocean water
$\Phi_{T,S}$		non-dimensional change of $\{T,S\}$ over boundary layer
$\Phi_{T,S,\mathrm{ref}}$		non-dimensional reference value for change of $\{T,S\}$
Φ_{turb}		non-dimensional change of $\{T,S\}$ due to turbulence
$\Phi_{T,S}^{\mathrm{mol}}$		non-dimensional change of $\{T,S\}$ due to molecular processes
ρ_i		density of ice at interface
ρ_o		density of ocean water at interface
ξ_N	$=$	$0.052,$ dimensionless universal constant
ζ	$=$	$fn/u_*\eta_*,$ non-dimensional distance from interface (positive into ice)
ζ_0		non-dimensional surface roughness
ζ_{ref}		reference non-dimensional distance from interface

2 Forcing Points in the Immersed Boundary Method

The Immersed Boundary Method (IBM) used to represent the boundary between the ice shelf and the ocean in the Parallel Ocean Program (POP) can apply forcing at grid points adjacent to the boundary location that are either just exterior to the fluid (ghost points) or just interior to the fluid (band points). Though ghost points have the desirable property that the forcing does not directly modify the fluid in the "real" portion of the computational domain, but only the "fictitious" portion that is simulated within the solid body (the ice shelf). Nevertheless, Choi et al. (2007) opted to use band points when representing a turbulent boundary layer because this allowed the modification of the flow to closely mimic the so-called "log law" for mean velocity near the solid boundary. A qualitatively similar boundary layer exists in the vicinity of the ice shelf/ocean interface. For this reason, I have opted to use the band point forcing, interior to the "real" fluid domain in my IBM. For each band point, I first use an interpolation method to compute the value of the fluid properties at an image point that lies deeper in the fluid (Choi et al., 2007). Then I interpolate values for the fluid properties at each band point using the image and boundary values.

3 Tangential velocity boundary layer solution

McPhee (1981) proposed an analytic solution for the mean tangential velocity (mean in the sense of the Reynolds average) within the turbulent boundary layer below the ice/ocean interface. The velocity solution is broken into two parts, one for the sublayer in which the eddy viscosity varies with distance from the interface and where viscous and roughness effects play a role, and one for the outer layer in which the eddy viscosity can be considered to be constant. The velocity solution is represented as a complex number, where the real part is the x component and the imaginary part is the y component. The non-dimensional form of the solution is (McPhee, 1981, Eq. (17)):

$$\hat{\boldsymbol{u}} = \begin{cases} -i\delta e^{\delta\zeta} & \zeta \leq -\xi_N \\ -i\delta e^{-\delta\xi_N} - \frac{\eta_*}{k}\left[\ln\frac{|\zeta|}{\xi_N} + (\delta - a)(\zeta + \xi_N) - \frac{a}{2}\delta(\zeta^2 - \xi_N^2)\right], & \zeta > -\xi_N \end{cases}, \quad (1)$$

where $k = 0.4$ and $\xi_N = 0.052$ are universal constants, where $\delta = (\pm i/k\xi_N)^{1/2}$ (positive in the northern hemisphere, negative in the southern), where $\hat{\boldsymbol{u}} = (\boldsymbol{u}_t - \boldsymbol{u}_{t,\infty})/\boldsymbol{u}_*$ and $\zeta = fn/u_*\eta_*$, and where $\boldsymbol{u}_*$ is the friction velocity (with magnitude equal to the square root of the magnitude of the kinematic stress) at the interface, $\boldsymbol{u}_{t,\infty}$ is the velocity outside the boundary layer, f is the local Coriolis parameter, and n is the distance from the interface in the direction normal to the interface (negative into the ocean, so that if the interface is horizontal, $n = z$, the usual height above sea level). The effects of buoyancy are parameterized in terms of L, the Obukhov length scale, or non-dimensional parameters μ_*, η_* and a involving this length:

$$L \equiv \frac{\rho_0 u_*^3}{gk\left(\gamma_S \left\langle u_z' S' \right\rangle_0 - \gamma_T \left\langle u_z' T' \right\rangle_0\right)}, \tag{2}$$

$$\mu_* \equiv \frac{u_*}{fL}, \tag{3}$$

$$\eta_* \equiv \left(1 + \frac{\xi_N \mu_*}{R_c}\right)^{-1/2}, \tag{4}$$

$$a \equiv \left(\frac{1}{R_c} + \frac{1}{\mu_* \xi_N}\right)(1-\eta*)\,\mu_* \eta_*, \tag{5}$$

where γ_S and γ_T are the expansion coefficients for salinity and temperature, respectively, and where $R_c \approx 0.2$ is the critical Richardson number. The Reynolds averaged vertical salinity flux $\left\langle u_z' S' \right\rangle_0$ and temperature flux $\left\langle u_z' T' \right\rangle_0$ will be discussed in Sec. 4, where methods are given for computing these values in terms of the bulk fluid properties.

Values for $\boldsymbol{u}_{t,\infty}$ and $\boldsymbol{u}_*$ can be found from the velocity $\boldsymbol{u}_t(n_{\mathrm{ref}})$ at some reference height n_{ref} (with corresponding non-dimensional value ζ_{ref}), and the fact that $\boldsymbol{u}_t(n_0) = 0$, where n_0 is the roughness length scale with corresponding non-dimensional value ζ_0:

$$\begin{aligned} \hat{\boldsymbol{u}}(\zeta_{\mathrm{ref}}) &= \frac{\boldsymbol{u}_t(n_{\mathrm{ref}}) - \boldsymbol{u}_{t,\infty}}{\boldsymbol{u}_*} \\ &= (-i\delta)e^{\delta\zeta_{\mathrm{ref}}} \end{aligned} \tag{6}$$

$$\begin{aligned} \hat{\boldsymbol{u}}_0 &= \frac{\boldsymbol{u}_t(n_0) - \boldsymbol{u}_{t,\infty}}{\boldsymbol{u}_*} \\ &= -\frac{\boldsymbol{u}_{t,\infty}}{\boldsymbol{u}_*} \\ &= \left\{-i\delta e^{-\delta\xi_N} - \frac{\eta_*}{k}\left[\ln\frac{|\zeta_0|}{\xi_N} + (\delta - a)\xi_N + \frac{a}{2}\delta\xi_N^2)\right]\right\}, \end{aligned} \tag{7}$$

where we take the outer solution for $\hat{\boldsymbol{u}}$ regardless of the value of ζ_{ref} in Eq. (6), and where we have assumed that $|\zeta_0| \ll \xi_N$ in Eq. (7). Wall roughness is more commonly expressed in terms of a drag coefficient C_D, rather than as a roughness length scale. Eq. (7) is dominated by the term involving ζ_0 in most cases, and this is the only term considered for the so-called quadratic drag formulation. Under these assumptions, the scalar version of Eq. (7), in the case of neutral buoyancy ($\eta_* = 1$), reduces to

$$u_{t,\infty} \approx -u_* \frac{1}{k} \ln\frac{|\zeta_0|}{\xi_N}. \tag{8}$$

Comparison with the quadratic drag relation,

$$u_*^2 \equiv C_D u_{t,\infty}^2, \tag{9}$$

allows us to relate ζ_0 to C_D:

$$C_D = \left(-\frac{1}{k}\ln\frac{|\zeta_0|}{\xi_N}\right)^{-2}, \tag{10}$$

$$\zeta_0 = -\xi_N e^{-k/\sqrt{C_D}}. \tag{11}$$

3.1 Iterative algorithm

Given values for n_{ref} and $\boldsymbol{u}_{t,\text{ref}} = \boldsymbol{u}_t(n_{\text{ref}})$ (and assuming fixed η_* and a), we need an algorithm for finding $\boldsymbol{u}_{t,\infty}$ and $\boldsymbol{u}_*$. A successful iterative algorithm that converges quickly (typically in less than ten iterations) is the following:

$$\hat{\boldsymbol{u}}_0 = \left\{-i\delta e^{-\delta\xi_N} - \frac{\eta_*}{k}\left[\ln\frac{|\zeta_0|}{\xi_N} + (\delta - a)\xi_N + \frac{a}{2}\delta\xi_N^2)\right]\right\}$$

$$\boldsymbol{u}_{t,\infty}^0 = \boldsymbol{u}_{t,\text{ref}}$$

$$\boldsymbol{u}_*^0 = -\frac{\boldsymbol{u}_{t,\infty}^0}{\hat{\boldsymbol{u}}_0}$$

for $k = 1, 2, \ldots k_{\text{max}}$

$$\quad \zeta_{\text{ref}}^{k-1} = \frac{n_{\text{ref}} f}{\eta_* |\boldsymbol{u}_*^{k-1}|}$$

$$\quad \boldsymbol{u}_*^k = -\frac{u_{\text{ref}}}{\hat{\boldsymbol{u}}_0}\frac{1}{1 + \frac{i\delta}{\hat{\boldsymbol{u}}_0} e^{\delta\zeta_{\text{ref}}^{k-1}}}$$

$$\quad \varepsilon = \frac{\left|\boldsymbol{u}_*^k - \boldsymbol{u}_*^{k-1}\right|}{\left|\boldsymbol{u}_*^k\right|}$$

  if $\varepsilon < 10^{-6}$

    $k_{\text{final}} = k$

    break

  end if

end for

$$\boldsymbol{u}_{t,\infty}^{k_{\text{final}}} = -\boldsymbol{u}_*^{k_{\text{final}}} \hat{\boldsymbol{u}}_0$$

4 Coupled temperature and salinity boundary layer solution

The boundary layer structure of temperature and salinity are similar to those found in McPhee et al. (1987) and in Holland and Jenkins (1999). The so-called three equations, those for freezing temperature of sea water, heat flux and salt flux, must be simultaneously satisfied at the wall:

$$T_0 = a_T S_0 + b_T + c_T p_0(z_0), \tag{12}$$

$$Q_i^T - Q_o^T = Q_{\text{latent}}^T, \tag{13}$$

$$Q_i^S - Q_o^S = Q_{\text{brine}}^S, \tag{14}$$

where z_0 is the height (negative below sea level) of the interface. Subscripts i and o represent ice and ocean properties, respectively, while subscript 0 represents quantities at the interface. Equation (12) is a linearization of the freezing point valid for salinity in the range 4–40 psu. Equation (13) can be expanded as

$$-\rho_i c_{p,i} \kappa_i^T \frac{(T_i - T_0)}{\Delta n_i} - \rho_o c_{p,o} \left\langle u_n' T' \right\rangle_0 = -\rho_o u_{n,0} L_f, \tag{15}$$

where ρ_i and ρ_o are the densities of ice and ocean water, respectively, at the interface, $c_{p,i}$ and $c_{p,o}$ are the specific heat capacities of ice and ocean water, respectively, L_f is the latent heat of fusion, and $u_{n,0}$ is the melt rate in the ocean (related to the ice melt rate by mass continuity, $\rho_o u_{n,0} = \rho_i u_{n,0,i}$). I have assumed that the temperature flux of ice can be parameterized in terms of its molecular diffusivity, κ_i^T, and using some reference temperature $T_i = T_i(\Delta n_i)$ a distance Δn_i above the ice/ocean interface. Similarly, the equation for salt flux can be reduced to

$$-\rho_o \left\langle u_n' S' \right\rangle_0 = -\rho_o u_{n,0} S_0, \tag{16}$$

where I have assumed that the salinity of the ice shelf is zero for all time (and therefore that the salt flux into the ice is also zero). This assumption will not be valid when frazil ice forms under ice shelves, but this process is thought to occur predominantly outside the ocean boundary layer, with saline ice being driven upward toward the ice/ocean interface by buoyancy (Holland and Jenkins, 1999). Therefore, the process of ice formation with nonzero salinity will not be considered within the boundary layer formulation.

Following McPhee et al. (1987), we can express the Reynolds averaged turbulent heat and salinity fluxes in terms of diffusion of bulk temperature and salinity normal to the interface,

$$\left\langle u_n' T' \right\rangle_0 = \frac{\rho_i}{\rho_o} \frac{c_{p,i}}{c_{p,o}} \frac{\kappa_i^T}{\Delta n_i} (T_i - T_0) + u_{n,0} \frac{L_f}{c_{p,o}} = -K_h \frac{\partial T}{\partial n}, \tag{17}$$

$$\left\langle u_n' S' \right\rangle_0 = u_{n,0} S_0 = -K_S \frac{\partial S}{\partial n}, \tag{18}$$

where K_h and K_S are the turbulent plus molecular diffusivities for heat and salinity, respectively, analogous to the eddy viscosity commonly used in closures for the Reynolds averaged momentum equation. Equations (17) and (18) can be non-dimensionalized and then integrated from the interface to an arbitrary normal distance n to obtain

$$\frac{T(n)-T_0}{\langle u_n'T'\rangle_0/u_*} = \Phi_T(n) = \int_n^0 \frac{u_* dn'}{K_h}, \tag{19}$$

$$\frac{S(n)-S_0}{\langle u_n'S'\rangle_0/u_*} = \Phi_S(n) = \int_n^0 \frac{u_* dn'}{K_S}. \tag{20}$$

I will assume that $T(n)$ and $S(n)$ have known values T_{ref} and S_{ref} at some reference distance from the interface n_{ref}, as I did for the tangential velocity in Sec. 3, and that $\Phi_{T,S}$ are known at this same reference distance (see below). With this assumption, I eliminate the unknown quantities T_0 and $u_{n,0}$ in Eqs (12), (19) and (20) leaving an equation in a single unknown, S_0, the salinity at the interface:

$$0 = c_2 S_0^2 + c_1 S_0 + c_0, \tag{21}$$

$$c_2 = a_T(d_1 - 1), \tag{22}$$

$$c_1 = (T_{\mathrm{ref}} - b_T - c_T p_0) - d_1(T_i - b_T - c_T p_0) + d_2, \tag{23}$$

$$c_0 = -d_2 S_{\mathrm{ref}}, \tag{24}$$

$$d_1 = \Phi_{T,\mathrm{ref}} \frac{\rho_i}{\rho_o} \frac{c_{p,i}}{c_{p,o}} \frac{\kappa_i^T}{\Delta n_i u_*}, \tag{25}$$

$$d_2 = \frac{\Phi_{T,\mathrm{ref}}}{\Phi_{S,\mathrm{ref}}} \frac{L_f}{c_{p,o}}. \tag{26}$$

Equation (21) can be solved using the quadratic formula. If only one real, positive root exists, I take this to be the solution. Alternative methods for finding S_0 may be required to handle cases where Eq. (21) has either zero or two real, positive solutions. Once a solution for S_0 has been found, values for $u_{n,0}$ and T_0 can be computed from Eqs. (12), (18) and (20)

$$T_0 = a_T S_0 + b_T + c_T p_0, \tag{27}$$

$$u_{n,0} = u_* \frac{(S_{\mathrm{ref}} - S_0)}{\Phi_{S,\mathrm{ref}} S_0}. \tag{28}$$

The remaining task is to specify the functional form for $\Phi_{S,T}(n)$. This is accomplished as in McPhee (1983) and McPhee et al. (1987) by assuming that the salinity and heat fluxes fall off linearly from their surface values to zero at the edge of the boundary layer

$$\langle u_n'\lambda'\rangle = -K\frac{\partial \lambda}{\partial n} = \langle u_n'\lambda'\rangle_0 \left(1 - \frac{n}{n_{\mathrm{bl}}}\right), \tag{29}$$

$$n_{\mathrm{bl}} = -k u_* \eta_* / f, \tag{30}$$

where λ is one of T or S. The two papers differ slightly on how they assume the eddy viscosity (assumed to be the same as the turbulent diffusivity for both salinity and temperature) varies within the surface layer, and therefore how thick the surface layer is. McPhee et al. (1987) assumes the eddy viscosity is linear within the surface layer and constant within the outer layer

$$K = \begin{cases} -knu_* & n_0 > n \geq n_{\mathrm{sl}}, \\ -kn_{\mathrm{sl}}u_* & n_{\mathrm{sl}} > n \geq n_{\mathrm{bl}}, \end{cases} \tag{31}$$

$$n_{\mathrm{sl}} = -u_* \eta_*^2 \xi_N / f. \tag{32}$$

With this definition, Eq. (29) can be integrated with respect to n. For n outside the surface layer, $n > n_{\mathrm{sl}}$,

$$\begin{aligned} \frac{\lambda(n) - \lambda_0}{\langle u_n' \lambda' \rangle_0 / u_*} &= \Phi_{\mathrm{turb}}(n) \\ &= -\frac{1}{k} \int_{n_{\mathrm{sl}}}^{n_0} \left(\frac{1}{n'} - \frac{1}{n_{\mathrm{bl}}} \right) dn' - \frac{1}{kn_{\mathrm{sl}}} \int_{n}^{n_{\mathrm{sl}}} \left(1 - \frac{n'}{n_{\mathrm{bl}}} \right) dn' \end{aligned} \tag{33}$$

In the limit that $|n_0| \ll |n_{\mathrm{sl}}|$, this integral is

$$\Phi_{\mathrm{turb}}(n) = \frac{1}{k} \ln \frac{n_{\mathrm{sl}}}{n_0} - \frac{1}{k} + \frac{n}{kn_{\mathrm{sl}}} - \frac{n^2}{2kn_{\mathrm{sl}}n_{\mathrm{bl}}}. \tag{34}$$

McPhee et al. (1987) argues that molecular fluxes must also be taken into account

$$\Phi_{T,S}^{\mathrm{mol}} = b \left(\frac{u_* n_0}{\nu} \right)^{1/2} \left(\frac{\nu}{\kappa_o^{T,S}} \right)^{2/3}, \tag{35}$$

where ν is the viscosity of ocean water, κ_o^T and κ_o^S are the molecular diffusivities of temperature and salinity, respectively, and where $b = 1.57$ is a universal constant found by fit to observations. The total non-dimensional change in temperature and salinity from the surface to a distance n is the sum of Eqs. (34) and (35),

$$\Phi_{T,S}(n) = \Phi_{\mathrm{turb}}(n) + \Phi_{T,S}^{\mathrm{mol}}. \tag{36}$$

Since the buoyancy flux is directed vertically upward whereas the surface normal need not be vertical (though it will, in general, be close to vertical because of the large horizontal to vertical aspect ratio of the system), it is necessary to specify the relationship between $\langle u_n' \lambda' \rangle_0$ and $\langle u_z' \lambda' \rangle_0$ (where, again, $\lambda = \{T, S\}$), in order to compute the Obukhov length, Eq. (2). It seems reasonable to assume that the impact of buoyancy on the shape of the boundary layer normal to the interface will go to zero as the interface becomes vertical ($L \to \infty$, $\mu_* \to 0$, $\eta_* \to 1$ and $a \to 0$). Since

$$u_z' = \boldsymbol{u}' \cdot \hat{\boldsymbol{z}} = u_t' \hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{z}} + u_n' \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}, \tag{37}$$

this suggests that $\langle u_t' \lambda' \rangle_0$ contributes negligibly to the vertical fluxes, so that the vertical flux of temperature and salinity is

$$\langle u_z' \lambda' \rangle_0 = \langle u_n' \lambda' \rangle_0 n_z, \tag{38}$$

where $n_z = \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}$ is the vertical component of the unit normal vector pointing from the ocean into the ice. Equation Eq. (2) becomes

$$L = \frac{\rho_0 u_*^3}{gkn_z \left(\gamma_S \langle u_n' S' \rangle_0 - \gamma_T \langle u_n' T' \rangle_0\right)}, \tag{39}$$

where $\langle u_n' T' \rangle_0$ and $\langle u_n' S' \rangle_0$ are computed from Eqs. (19) and (20), respectively.

Computation of $\boldsymbol{u}_*$ and $\boldsymbol{u}_\infty$ requires knowledge of L, and therefore of T_0 and S_0. Since computations of T_0 and S_0 themselves involve u_*, it is necessary to solve for all four of these parameters simultaneously using and iterative method, as proposed in McPhee et al. (1987). The iterative procedure begins by assuming neutral stability, giving values for $\boldsymbol{u}_*$ and $\boldsymbol{u}_\infty$ that do not depend on T_0 and S_0. Using this value for u_0, T_0 and S_0 can be computed. The results are used to compute a new value of L, μ_*, η_* and a. The process is repeated by obtaining new values of $\boldsymbol{u}_*$ and $\boldsymbol{u}_\infty$, and so on until a convergence criterion is met.

5 Interdependence of values at image points and band points

The picture becomes somewhat more complicated when the boundary layer solutions are applied within the IBM. We do not, in general, know the value of velocity, temperature or salinity at image points (the same as the values at n_{ref} in the previous sections) independently of their values at the band points. This is because the value of a property at the image point is found by interpolation from neighboring value, including those at the band point. Since the value at the band point depends on the value at the image point, the we must simultaneously solve for both the image and band values.

The image value is found by interpolation (the linear sum of values at neighboring points times weights), which may include the band point. In what follows, we will assume the image point to be at a distance $2n$ and the band point to be a distance n from the boundary. Velocity, temperature and salinity at the image point are related to their respective values at the band point by the relations

$$\boldsymbol{u}_t(2n) = \sum_{j=1}^{N-1} w_j \boldsymbol{u}_{t,j} + w_N \boldsymbol{u}_t(n), \tag{40}$$

$$T(2n) = \sum_{j=1}^{N-1} w_j T_j + w_N T(n), \tag{41}$$

$$S(2n) = \sum_{j=1}^{N-1} w_j S_j + w_N S(n), \tag{42}$$

where the sums are over nearby neighbors that are in the fluid but that are *not* band points. Making use of Eqs. (6) and (7), where the former is evaluated at $n_{\text{ref}} = 2n$ and $n_{\text{ref}} = n$, we have

$$\boldsymbol{u}_{t,\infty} = -\hat{\boldsymbol{u}}_0 \boldsymbol{u}_* \tag{43}$$

$$\boldsymbol{u}_t(2n) = \boldsymbol{u}_{t,\infty} - \boldsymbol{u}_* i\delta e^{\delta 2nf/|u_*|\eta_*} \tag{44}$$

$$\boldsymbol{u}_t(n) = \boldsymbol{u}_{t,\infty} - \boldsymbol{u}_* i\delta e^{\delta nf/|u_*|\eta_*} \tag{45}$$

Equations (40), (43), (44) and (45) can be solved simultaneously using an iterative scheme similar to the one in Sec. 3.1. The term used to compute $\boldsymbol{u}_*^k$ in terms of ζ_{ref}^{k-1} is replaced by

$$\boldsymbol{u}_*^k = -\frac{\sum_{j=1}^{N-1} w_j \boldsymbol{u}_{t,j}}{\hat{\boldsymbol{u}}_0} \frac{1}{(1-w_N) + \frac{i\delta}{\hat{\boldsymbol{u}}_0}\left(e^{\delta 2nf/|u_*^{k-1}|\eta_*} - w_N e^{\delta nf/|u_*^{k-1}|\eta_*}\right)}. \tag{46}$$

Note that, if $w_N = 0$, then the term involving the sum over j is equal to $\boldsymbol{u}_t(2n) = \boldsymbol{u}_{t,\text{ref}}$, and we recover the same term as in Sec. 3.1. In general, $w_N \ll 1$, so that the terms involving w_N will be small perturbations to the earlier scheme, not expected to affect its convergence properties.

Incorporating Eqs. (41) and (42) into the solutions for temperature and salinity from Sec. 4 is a bit messier, but we will see that the result is still a quadratic equation for S_0, the salinity at the interface, whose solution can be used to compute the melt rate, interface temperature, and temperatures and salinities at the image and band points. These equations, together with Eqn. (12) for the freezing temperature and Eqns. (19) and (20) evaluated at $n_{\text{ref}} = 2n$ and $n_{\text{ref}} = n$ lead to seven total equations in the seven unknowns S_0, T_0, $u_{n,0}$, $T(2n)$, $T(n)$, $S(2n)$ and $S(n)$. The seven equations can be written as follows

$$S(2n) = c_0 + c_1 S(n), \tag{47}$$

$$T(2n) = c_2 + c_3 T(n), \tag{48}$$

$$T(n) = c_4 + c_5 T_0 + c_6 u_{n,0}, \tag{49}$$

$$T(2n) = c_7 + c_8 T_0 + c_9 u_{n,0}, \tag{50}$$

$$T_0 = c_{10} + c_{11} S_0, \tag{51}$$

$$S(n) = c_{12} S_0 + c_{13} u_{n,0} S_0, \tag{52}$$

$$S(2n) = c_{14} S_0 + c_{15} u_{n,0} S_0, \tag{53}$$

where the constants c_n are given by

$$c_0 = \sum_{j=1}^{N-1} w_j S_j, \tag{54}$$

$$c_1 = w_N, \tag{55}$$

$$c_2 = \sum_{j=1}^{N-1} w_j T_j, \tag{56}$$

$$c_3 = w_N, \tag{57}$$

$$c_4 = b_0 \Phi_T(n) T_i, \tag{58}$$

$$c_5 = 1 - b_0 \Phi_T(n), \tag{59}$$

$$c_6 = \frac{L_f}{c_{p,o}} \Phi_T(n), \tag{60}$$

$$c_7 = b_0 \Phi_T(2n) T_i, \tag{61}$$

$$c_8 = 1 - b_0 \Phi_T(2n), \tag{62}$$

$$c_9 = \frac{L_f}{c_{p,o}} \Phi_T(2n), \tag{63}$$

$$c_{10} = b_T + C_T p_0, \tag{64}$$

$$c_{11} = a_T, \tag{65}$$

$$c_{12} = 1, \tag{66}$$

$$c_{13} = \frac{\Phi_S(n)}{u_*}, \tag{67}$$

$$c_{14} = 1, \tag{68}$$

$$c_{15} = \frac{\Phi_S(2n)}{u_*}, \tag{69}$$

and where

$$b_0 = \frac{\rho_i}{\rho_o} \frac{c_{p,i}}{c_{p,o}} \frac{\kappa_i^T}{\Delta n_i u_*}. \tag{70}$$

All unknowns except for S_0 can be eliminated from Eqs. (47)–(53):

$$0 = e_0 + e_1 S_0 + e_2 S_0^2, \tag{71}$$

where

$$e_0 = d_2 d_3, \tag{72}$$

$$e_1 = d_2 d_4 - d_0 d_5, \tag{73}$$

$$e_2 = -d_1 d_5, \tag{74}$$

$$d_0 = -c_2 - c_3 c_4 - c_3 c_5 c_{10} + c_7 + c_8 c_{10}, \tag{75}$$

$$d_1 = -c_3 c_5 c_{11} + c_8 c_{11}, \tag{76}$$

$$d_2 = -c_3 c_6 + c_9, \tag{77}$$

$$d_3 = -c_0, \tag{78}$$

$$d_4 = -c_1 c_{12} + c_{14}, \tag{79}$$

$$d_5 = -c_1 c_{13} + c_{15}. \tag{80}$$

We solve Eq. (71) using the quadratic formula. If the roots are complex or negative, an alternative formulation of the problem is required. If one root is positive while the other is not, the positive root is the physical solution. For the time being, if both roots are positive, the smaller root is assumed to be the physically real root. (We need a more sophisticated method that takes into account whether melting or freezing is occurring!). Then, the melt rate is given by

$$u_{n,0} = \frac{-d_0 - d_1 S_0}{d_2} \tag{81}$$

We find T_0 using Eq. (51), $T(n)$ using Eq. (49), $T(2n)$ using Eq. (50), $S(n)$ using Eq. (52), and $S(2n)$ using Eq. (53).

6 Normal velocity boundary condition

The normal component of velocity at the interface is equal to the melt rate of the ice. This melt rate is generally very small (on the order of tens to hundreds of meters per year) and can probably be taken to be zero for the purposes of computing the velocity boundary condition normal to the interface. We compute the normal component of the velocity at a given band point using linear interpolation between the boundary value and the value at the image point,

$$u_n(n) = \frac{1}{2}(u_n(2n) + u_{n,0}). \tag{82}$$

The total velocity at a band point a distance n from the interface is

$$\boldsymbol{u}(n) = u_{t,1}(n)\hat{\boldsymbol{t}}_1 + u_{t,2}(n)\hat{\boldsymbol{t}}_2 + u_n(n)\hat{\boldsymbol{n}}. \tag{83}$$

From this expression, the horizontal and vertical velocity components of the velocity can be computed for band points on the U and T grids, respectively.

A complication is that the velocity must remain divergence free. Since the geometry of the interface is specified using a height field (and, therefore, there is only one boundary intersection per vertical column), we can use the barotropic momentum and continuity equations to insure that the flow simultaneously remains divergence free and satisfies the boundary conditions on the vertical velocity. This can be accomplished by solving the rigid lid barotropic equations only in the region of "real" flow, between $z = -b$ and $z = -H$. The modified version of Eq. (126) from Smith and Gent (2002), the elliptic equation for the sea surface height η^{n+1} (where the sea surface height can be related to the surface pressure by $p_s = \rho_0 g \eta$) is given by

$$\nabla \cdot (H-b)\nabla\eta^{n+1} = \nabla \cdot (H-b)\left[\frac{\hat{\boldsymbol{U}}}{g\alpha\tau} + \nabla\eta^{n-1}\right] + \frac{w_b}{g\alpha\tau}, \tag{84}$$

where $\hat{\boldsymbol{U}}$ is the auxiliary velocity as defined in Eq. (124) of Smith and Gent (2002), $\alpha = 1/3$, $\tau = 2\Delta t$ is twice the time step, and where the vertical velocity at the interface w_b is given by

$$w_b = \boldsymbol{u}(x,y,z=-b(x,y)) \cdot \hat{\boldsymbol{z}}, \tag{85}$$

which is computed by linear extrapolation from the nearest ocean grid cell below $z=-b$ and an image point below that. (This choice of extrapolation produces $w_b = u_{n,0}$ in the case that the interface normal points vertically upward.)

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